Analysis and Control of Interconnected Positive Systems with Constraints

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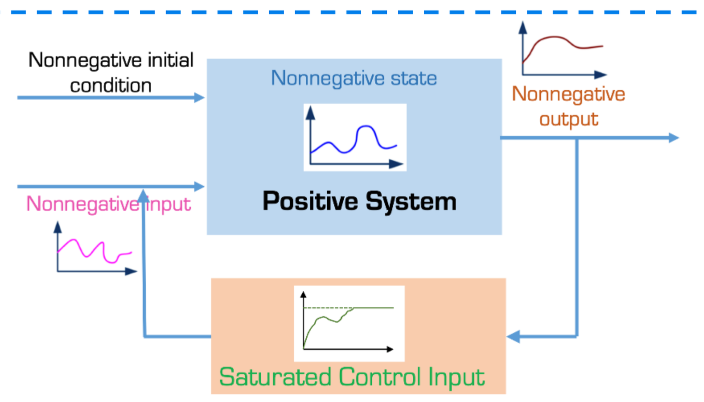
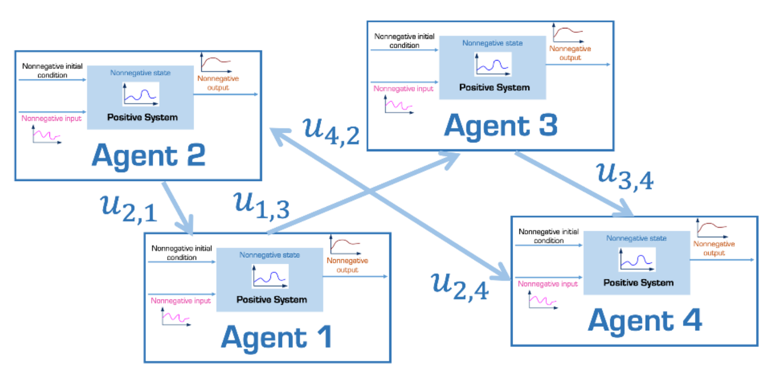
Introduction

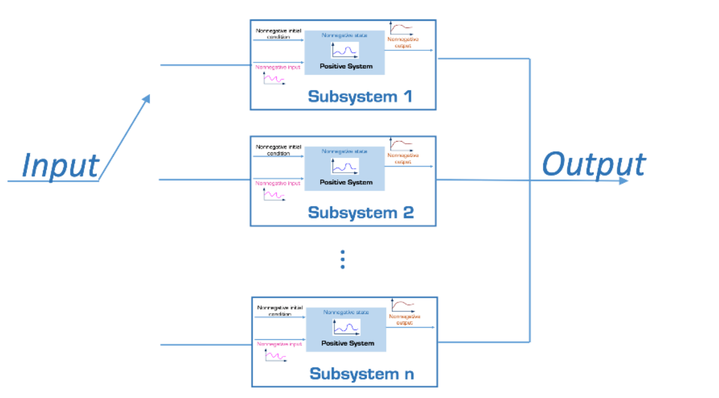
Interconnected positive systems with constraints are crucial in modeling and analyzing complex real-world processes, such as transportation networks, biological systems, and economic structures. These systems ensure that variables remain nonnegative, reflecting realistic limitations like resource availability or physical quantities. Understanding their importance enables researchers and engineers to design robust, stable, and reliable solutions that meet practical requirements while optimizing performance.



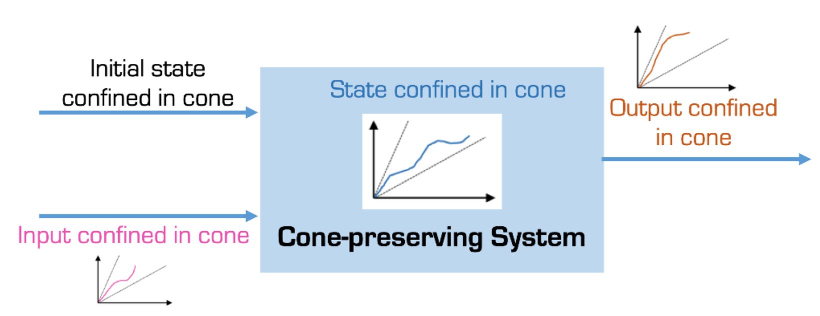
Saturated positive systems



Positive multi-agent systems



Periodically switched positive systems



Generalization of positive systems

Research 1: Estimation of domain of attraction

Saturated positive systems:

$$\dot{x}(t) = Ax(t) + B\mathrm{sat}(u(t))$$

$$\mathrm{sat}(u_i) = \begin{cases} 1 \text{ if } u_i \geq 1 \\ u_i \text{ if } u_i \in (-1,1) \\ -1 \text{ if } u_i \leq -1 \end{cases}$$

Domain of attraction with different Lyapunov functions

Invariant hyperpyramids:

$$P(\lambda,\rho) := \{x : x \succeq 0, \lambda^{\mathrm{T}} x \leq \rho\}$$

Sum-separable Lyapunov functions

$$V(x) = \lambda^{\mathrm{T}} x,$$

Characterization of the invariant set

$$\begin{aligned} &A + BD_j\mathbf{1}\lambda^{\mathrm{T}} + BD_j^- K \in \mathbb{M}_n, \\ &\tau_j - \lambda^{\mathrm{T}} BD_j\mathbf{1} \leq 0, \\ &\lambda^{\mathrm{T}} A - \tau_j \lambda^{\mathrm{T}} + \lambda^{\mathrm{T}} BD_j^- K \prec 0 \end{aligned}$$

Invariant hyperrectangles:

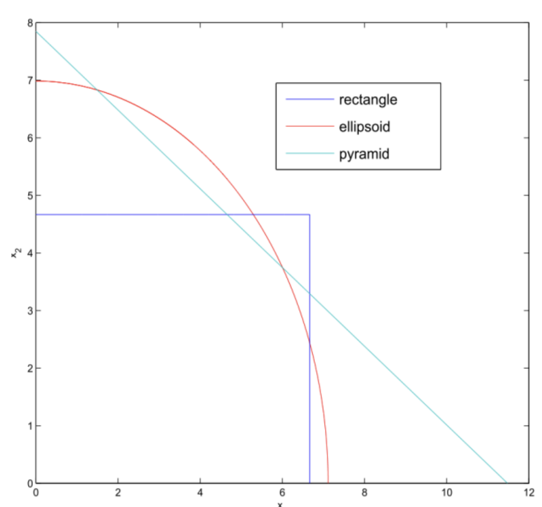
$$P := \mathrm{Rec}(x) = [0, x_1] \times [0, x_2] \times \cdots \times [0, x_n]$$

Max-separable Lyapunov functions

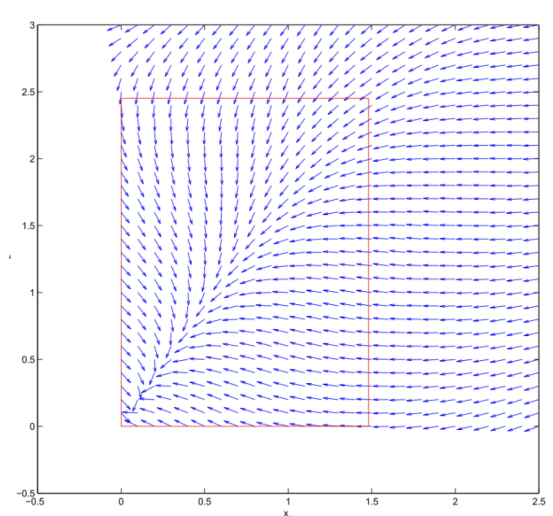
$$V(x) = \max_{1\leq i \leq n} |x_i|/\lambda_i = \left\|\mathrm{diag}(\lambda)^{-1} x\right\|_\infty$$

Characterization of the invariant set

$$\begin{aligned} &A + BD_j^- K + \Gamma_j \in \mathbb{M}_n, \\ &BD_j\mathbf{1} - \Gamma_j \lambda \succeq 0, \\ &\left(A + BD_j^- K + \Gamma_j\right)\lambda - BD_j\mathbf{1} \prec 0 \end{aligned}$$



Comparison of different domain of attractions



Normalized vector field with hyperrectangles

J. Shen and J. Lam, “Analysis of Positive Systems with Input Saturation: Invariant Hyper-Pyramids and Hyper-Rectangles,” IEEE Transactions on Automatic Control.

Research 2: Analysis of positive networked systems

Positive multi agent system:

Agent dynamic: $\dot{x}_i(t) = Ax_i(t) + Bu_i(t)$

Control law: $u_i(t) = K \sum_{v_j \in \mathcal{N}_i} a_{ij}(x_j(t) - x_i(t))$

Agent/edge consensus: $\lim_{t\to\infty} (x_j(t) - x_i(t)) = 0$

Positive consensus conditions:

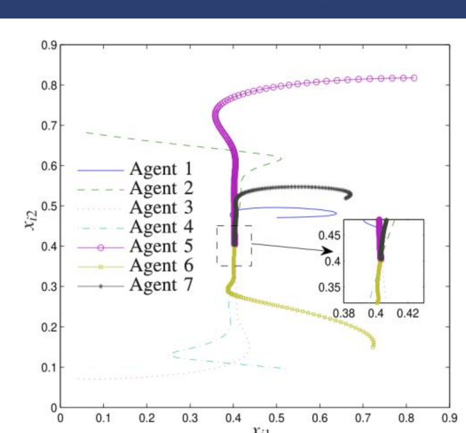
Structure of the control matrix $K = SB^{\mathrm{T}}P$ satisfying

$$\begin{aligned} &BSB^{\mathrm{T}}P \succeq 0, \\ &A - l_{max}BSB^{\mathrm{T}}P \text{ is Metzler}, \\ &A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}P < 0 \end{aligned}$$

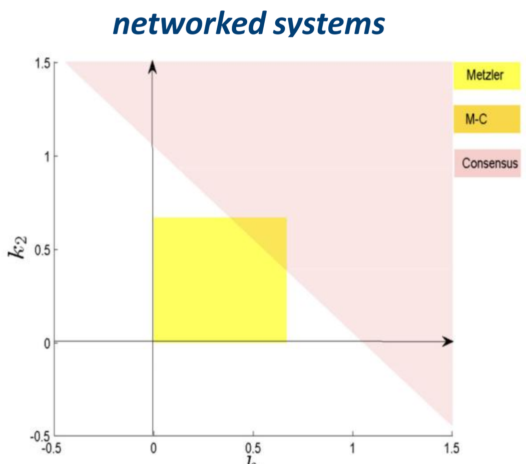
Non-negative edge consensus conditions:

Structure of the control matrix K satisfying

$$\begin{aligned} &BKQ \succeq 0, \\ &AQ - d_{max}BKQ \text{ is Metzler}, \\ &\begin{bmatrix} AG_i + G_i^{\mathrm{T}}A^{\mathrm{T}} - BKQK^{\mathrm{T}}B^{\mathrm{T}} & P_i^{\mathrm{T}} - G_i^{\mathrm{T}} + AH^{\mathrm{T}} & BKQ - \lambda_i G_i^{\mathrm{T}} \\ * & H_i - H_i^{\mathrm{T}} & -\lambda_i H_i \\ * & * & -Q \end{bmatrix} < 0 \end{aligned}$$



Consensus of positive networked systems



Controllable region

J. Liu, K.-W. Kwok, Y. Cui, J. Shen, and J. Lam, “Consensus of Positive Networked Systems on Directed Graphs,” IEEE Transaction on Neural Networks and Learning Systems.

J. Liu, J. Lam and K.-W. Kwok, “Further Improvements on Non-negative Edge Consensus of Networked Systems,” IEEE Transactions on Cybernetics.

Research 3: Analysis of periodic positive systems

Periodic piecewise positive systems without delays:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

where $A(t) = A_{\sigma(t)}$ and $B(t) = B_{\sigma(t)}$

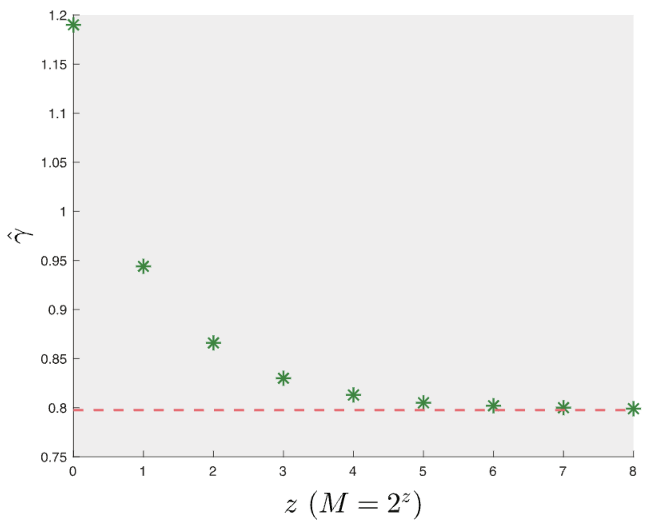
when $t \in \left[t_{i-1,\sigma(i)-1}, t_{i,\sigma(i)}\right)$

Periodic piecewise co-positive Lyapunov functions

$$V(t) = \sum_{i=1}^{m} \zeta_i(t) V_i(t)$$

where

$$\begin{aligned} V_i(t) &= x^T(t)p_i(t) \\ p_i &= p_{i-1} + \frac{T}{M}\varphi_i \\ \varphi_i &= \left(-\tilde{A}_M\right)^{M+1-j}(kq - A^{\mathrm{T}}p) \end{aligned}$$



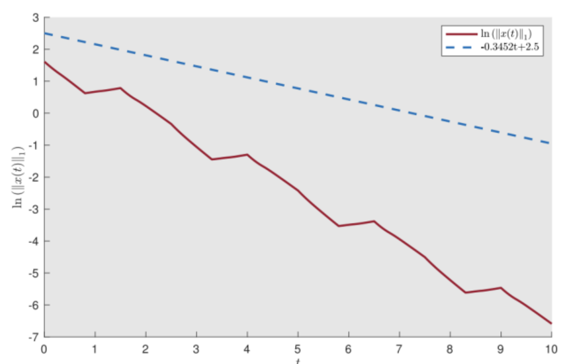
Precise characterization of convergent speed

Periodic piecewise positive systems with constant delays:

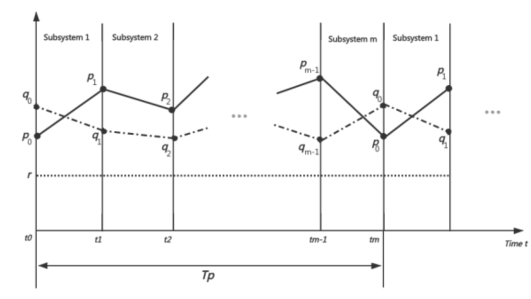
$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + A_d(t)x(t-d) + B_\omega(t)\omega(t) \\ z(t) &= C(t)x(t) + C_d(t)x(t-d) + D_\omega(t)\omega(t) \end{aligned}$$

Discontinuous co-positive Lyapunov-Krasovskii functional

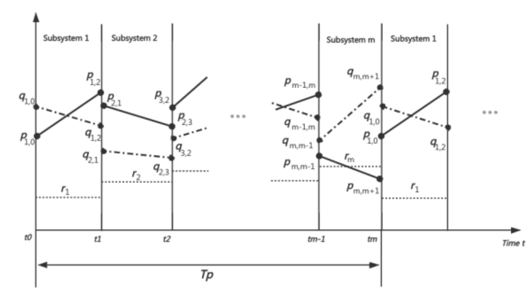
$$V_i(t) = e^{\lambda t}p_i^T(t)x(t) + \int_{t-d}^{t} e^{\lambda(\tau+d)}q_i^T(t)x(\tau)d\tau + \int_{-d}^{0}\int_{t+\tau}^{t} e^{\lambda(\theta+d)}r^T x(\theta)d\theta d\tau$$



Decay rate estimation



Continuous Lyapunov functional



Discontinuous Lyapunov functional

	λ	μ_1	μ_2	μ_3	γ	Upper bound
Theorem 2	N/A	N/A	N/A	N/A	8.300	8.300
Theorem 3	0.001	1	1	1	7.500	7.5192
	0.01	1	1	1	7.6380	7.8309
	0.01	1.01	1	1	7.4852	12.8764

Comparison of upper bound of the input-output gain

B. Zhu, J. Lam, X. Xie, X. Song, and K.-W. Kwok, “Stability and L1-gain Analysis of Periodic Piecewise Positive Systems with Constant Time Delay,” IEEE Transactions on Automatic Control.

Research 4: Analysis of cone-preserving systems

Cone Preserving systems:

$$\begin{aligned} x(k+1) &= Ax(k) + B_w w(k) \\ y(k) &= Cx(k) + D_w w(k) \end{aligned}$$

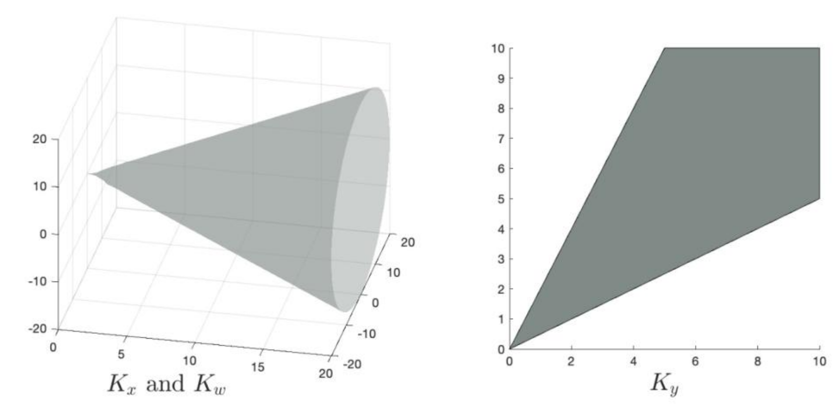
with $x(k) \in K_x$, $y(k) \in K_y$

Cone linear absolute-norm:

$$\|x\|_{\eta,1} = \inf\{\eta^T u : -u \preceq_K x \preceq_K u\}$$

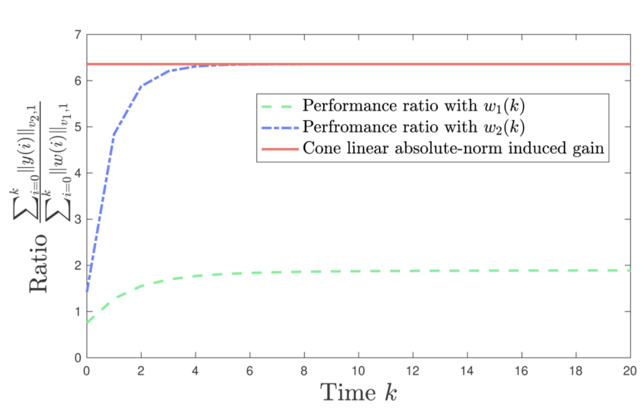
Cone max-norm:

$$\|y\|_{u,\infty} = \inf\{t \geq 0 | y \in tB_u\}$$

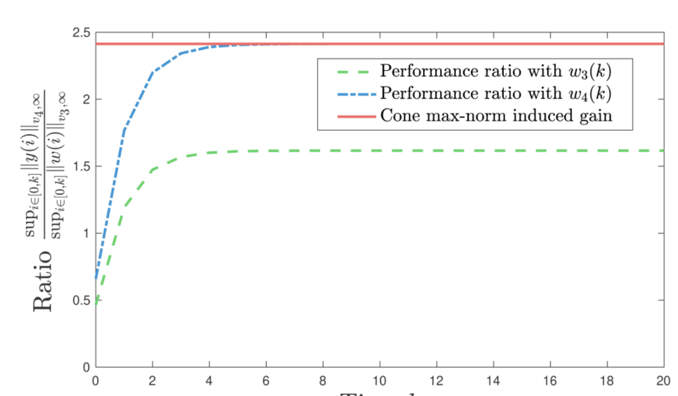


Demonstration of proper cone

Duality of the cone induced norm: The cone linear absolute-norm of the cone-preserving system is equal to the cone max-norm of the dual of the system, where v_1 and v_2 are the vectors of cone linear absolute-norm induced gain in system, where v_3 and v_4 are the vectors of cone-max norm induced gain in dual system.



Cone linear absolute-norm



Cone max-norm

B. Zhu, J. Lam, J. Shen, Y. Cui, X. Lu, and K.-W. Kwok, “Input-output Gain Analysis of Linear Discrete-time Systems with Cone Invariance,” IEEE Transactions on Automatic Control.

Conclusions

- **Domain of attraction:** Established non-conservative conditions for contractive invariance and highlighted the duality between different norm-based approaches in saturated positive systems.
- **Analysis of positive networked systems:** Provided the first analysis of nonnegative consensus tracking in positive networked systems and developed efficient and less conservative consensus and optimization methods.
- **Analysis of periodic positive systems:** Achieved less conservative bounds for input-output gains and constructed new co-positive Lyapunov functionals for periodically switched positive systems with delays.
- **Analysis of cone-preserving systems:** Extended stability analysis to cone-preserving systems, characterized cone-induced norms, and revealed duality of cone induced gains.